Modeling Fair Redistricting Commissions

Henry Fleischmann and Pingbang Hu

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1 Introduction

Every ten years, coinciding with the decennial census, all 50 American states redraw their voting maps. People migrate and accumulate, and our fundamental tenet of one person one vote must be re-established by ensuring equal population districts. However, lurking in the shadows of this triumphant symbol of our democracy is a pernicious and well-established evil: *gerrymandering*. Not all redistricting plans are created equal; when the redistricting process is commandeered by partisan actors drawing maps to maximize their electoral gains, the act is colloquially referred to as gerrymandering. In many states, elected politicians are able to participate in drawing the maps they will later run for office in. This amounts to politicians selecting their voters. It is perhaps not immediately obvious that it should even be possible to derive partisan advantage from merely deciding the boundaries of voting districts. A number of user-friendly simulations provide a convincing demonstration that it is indeed possible.[1]

In the past several years, there has been a growing movement to take the task of redistricting out of the hands of politicians. Fourteen states currently employ specially appointed *redistricting commissions* to draw some or all of their redistricting maps: Alaska, Arizona, Arkansas, California, Colorado, Hawaii, Idaho, Michigan, Missouri, Montana, New Jersey, New York, Ohio, Pennsylvania, Virginia, and Washington. Some commissions are composed entirely of non-politicians (Michigan and California, for example), while others are populated largely by career partisans (including New Jersey). These commissions must adhere to the idiosyncratic rules of their respective states, with one unifying objective being the design of redistricting maps that do not unduly advantage either political party.

While not without their imperfections, several non-partisan redistricting commissions have been lauded as a promising solution in the plight of reducing the incidence of partisan gerrymandering. For example, Princeton Gerrymandering's Redistricting Report Card and IPPSR's Partisan Advantage Tracker each suggest that Michigan and Colorado's redistricting plans yield minimal partisan lean. On the other hand, not all commissions enjoyed similar success. The New York redistricting commission was unable to find a consensus to decide on a plan, and ultimately the legislature drafted a Democratic gerrymander. This failure received widespread attention across the media spectrum.

The state constitutions delicately demarcate every function of these redistricting commissions. For example, Article XXI of the California State Constitution details the following aspects of the California Citizens Independent Redistricting Commissions:

[1] See *GerryMander* by the company GameTheory: `http://gametheorytest.com/gerry/`. Numerous other game-based demos exist.

- *Commission Composition:* The commission comprises exactly 14 members. This includes five members registered with each of the largest two political parties in California and four members not registered with either. Note that in California, you are asked to specify a partisan affiliation when registering to vote.
- *Commission Decision Protocol:* 9/14 votes are required for any official action of the commission, including three votes from each of the three groups of members. Once the commissioners decide on a redistricting plan, it goes directly to the Secretary of State for implementation. If that fails, the California Supreme Court appoints a special master to draw the maps.
- *Commissioner Selection:* Commissioners are selected in an effort to avoid infiltration by partisan actors. Numerous forms of relations to politicians are automatically disqualified. For example, commissioners cannot have, in the past 10 years: been appointed to, elected to, or a candidate for state or federal office; served as an officer, employee, or paid consultant of a political campaign or party; be a registered lobbyist; or contributed more than $2000 to any political candidate. Additionally, three independent auditors further screen candidates.

This language was added to Article XXI via two California Ballot Initiatives (Prop. 11, the Voters FIRST Act in 2008, and Prop. 20, the Voters FIRST Act for Congress in 2010). It also added a number of new requirements for legal maps, including the requirement of not favoring or disfavoring any political party.

These protocols appear to work well in California. However, the integrity of our democratic processes is essential, and it would be nice to provide more rigorous guarantees of the efficacy of redistricting commissions. In this work, we are most interested in the interplay between the composition of the commission and the protocol the commission uses for decision-making. Ultimately, each commission, regardless of its composition, is supposed to select a redistricting map that does not favor either major political party. It is natural to wonder about the following.

1. How should the commissioners decide on a map?
2. How does the mechanism for this decision depend on the composition of the commission?

These are the questions that guide our work.

2 Existing Work

In this work, we study strategy-proof voting rules that result in neutral maps. Perhaps the most well-known strategy-proof voting rule is the median voting rule [Bla48]. Black showed that, in single-peaked preference domains, the voting rule selecting the median preference is strategy-proof. This rule can be extended to restricted higher-dimensional preference domains.

One of the interesting aspects of our model is that strives to ensure a particular socially-desirable outcome (a neutral redistricting map) but does not know ahead of time the classification of the redistricting maps. Instead, we rely on the conflicting preferences of the commissioners to ensure this outcome. Other work has studied the effectiveness of private information on voting behavior in somewhat different settings. For example, in [FP97], they model the effect of an uncertain utility-affecting state variable on the behavior of voters in a two-candidate election.

One facet of redistricting commissioners is the potential for bargaining between groups with entirely contrasting objectives. One example might be unrelenting, hyperpartisan Democrat and

Republican commissioners trying to reach a compromise. The bargaining of this kind, without a neutral third party, has been the subject of considerable study in the past. See Nash's classic model of bargaining [Nas50], and later work permitting more players and the formation of coalitions [CJ10].

Quantitative methods have been applied to redistricting fairness in myriad ways. One recent line of work compares redistricting maps to randomly sampled "fair" plans, concluding that the plan is unfair if its partisan outcomes differ dramatically from statistics of the sampled map [CFMP20, DDS21, ACH+20, ACH+21]. This is a recent and dramatic improvement over some of the previously existing methods. Duchin gives an excellent survey [Duc18].

Others have tried to directly apply ideas from social choice theory to combat gerrymandering. One recent work analyzed multi-member districts as a way of combatting partisan gerrymandering [GGRS22].

3 The Model

We wish to model the function of a redistricting commission selecting a redistricting map from a collection of candidate maps. In many states, the commission is composed of some known number of Democratic voters, Republican voters, and Independent or unaffiliated voters. Given the nature of the commission, we propose that these commissioners can be classified into a few different subclasses. First, the D commissioners prefer any map that does not advantage Republicans over any map that does advantage Republicans. However, some unknown number of the D commissioners prefer neutral maps over those that advantage Democrats. That is, the D commissioners are partitioned into two types, w_D and s_D: w_D signifies *weak Democrat*, commissioners who lean Democrat but prefer neutral maps over partisan maps; s_D instead denotes *strong Democrat*, commissioners who lean Democrat and prefer Democrat-advantaging maps over all others. We similarly define w_R and s_R for R commissioners who prefer any map that does not advantage j over any map that does advantage Democrats.

Given the polarized nature of American society, we assume that each I commissioner has a weak preference toward one party over the other. But, given that these commissioners do not self-identify a partisan affiliation and, in many selection mechanisms, will be chosen not to lean heavily toward either party, we assume that they prefer neutral maps over partisan maps. That is, the I commissioners are partitioned between those of type w_D and w_R. We assume that the exact distribution of commissioners among these subclasses is unknown.

For simplicity, we assume that the maps considered by the commission are of three types D, N, or R. We also assume that there is precisely one map of each type.[2] A map of type D advantages Democrats, a map of type N is neutral, and a map of type R advantages Republicans. We assume that each map type is known only to the commissioners. The intuition is that studying these maps is the full-time job of the commissioners for about a year, so by the end, they will possess unique insights. However, an outsider might be unable to evaluate maps effectively (nor should the mechanism depend on the set of candidate maps). Given these alternatives, we define a preference ordering for each of the four types of commissioners:

$$s_D : D \succ N \succ R, \quad w_D : N \succ D \succ R, \quad s_R : R \succ N \succ D, \quad w_R : N \succ R \succ D.$$

The goal of the commission is to select a neutral map. Hence, our goal is, given the number of D, R, and I commissioners and a collection of maps of unknown types, to construct a group

[2] This is justified in Remark 1.

strategy-proof social choice function mapping the collection of commissioner preferences to a neutral map. Our mechanism should also not permit stalemates resulting from a lack of consensus (an existing flaw of many commissions).

Remark 1. *Assuming that there is one map of each type is reasonably justified. Imagine having each group of commissioners propose a map. They are incentivized to propose a map of the type of their highest preference. It is possible that, for example, among the D commissioners that there are more w_D than s_D commissioners. In this case, they might choose to propose an additional map N_2 of type N. Suppose that the R commissioners still propose a map of type R. Then, the three proposed maps are N_1, N_2, and R. The commissioners are indifferent between N_1 and N_2, and so the commissioners preferring neutral maps can collaborate to all rank N_1 first. So, for all non-s_R commissioners their preference ordering will be $N_1 \succ N_2 \succ R$ and for all s_R commissioners their preference ordering will be $R \succ N_2 \succ N_1$. The mechanism in Theorem 4.2 then results in R receiving a score strictly less than 0 and N_1 receiving a score strictly greater than 0.*

4 Neutral Map Mechanisms

In this section, we introduce mechanisms for selecting neutral maps under this model. These mechanisms are strategy-proof in a stronger than usual sense: they are strategy-proof even upon allowing the joint defection of coalitions of commissioners of the same type (for example, all of the s_D commissioners jointly misreporting their preferences). This stronger notion of strategy-proofness is *group strategy-proofness*.

4.1 The Balanced Mechanism

We first consider a balanced mechanism, where there are equal numbers of Democrats and Republicans and at least one w_D or w_R commissioner. Theorem 4.1 shows that a natural positional-scoring voting rule achieves group strategy-proofness and chooses a neutral map.

Theorem 4.1. *Suppose that the commission is composed of an equal number of Democrats and Republicans and at least one Independent commissioner, w_D, or w_R. Then, the positional-scoring voting rule with scores $\langle 1, 0, -1\rangle$, respectively, is group strategy-proof and chooses a neutral map.*

Proof. Let the number of D, R, and I commissioners be given by the tuple (n, n, n'). Let

$$\begin{aligned} x &:= \#s_D, \quad n - x = \#w_D; \quad \text{(from } D\text{)} \\ y &:= \#s_R, \quad n - y = \#w_R; \quad \text{(from } R\text{)} \\ z &:= \#w_D, \quad n' - z = \#w_R. \quad \text{(from } I\text{)} \end{aligned}$$

First, we show that if all agents report their preferences truthfully, then N is selected. By truthful voting, the scores for D, R, and N are

- D: $x - n - n' + z \leq 0$;
- R: $y - n - z \leq 0$;
- N: $n - x + n - y + n' > 0$,

hence N indeed wins. Here we use the assumption that there is at least one w_D or w_R to say $n' > 0$.

Second, we prove that this mechanism is group strategy-proof by showing that there is no useful deviation for s_D's (hence nor for s_R's by symmetry).[3] To have a useful deviation for s_D's, we observe that the only possible useful deviation is misreporting $D \succ R \succ N$ since they need to ensure D to achieve a useful deviation.[4] Suppose that a group of x' s_D agents misreport their preference, for some $0 \leq x' \leq x$. Then, the resulting scores for alternatives D, R, and N are given by

- D': $x - n - n' + z$;
- R': $y + x' - n - z$;
- N': $2n - x - x' - y + n'$.

That is, $D' = D$, $R' = R + x'$, and $N' = N - x'$. Assume that $D' \geq R'$ since otherwise this is not a useful deviation. Then we have

$$x - n - n' + z \geq y + x' - n - z \Leftrightarrow x - x' - n' + 2z - y \geq 0$$

Finally, to have a useful deviation, we want $D' > N'$, i.e.,

$$x - n - n' + z > 2n - x - x' - y + n' \Leftrightarrow 2x + x' - 3n - 2n' + z + y > 0.$$

By adding $x - x' - n' + 2z - y \geq 0$, the above holds if

$$3x - 3n - 3n' + 3z > 0 \Leftrightarrow (x - n) + (z - n') > 0 \lightning$$

hence there are no useful deviations for s_D. □

One limitation of Theorem 4.1 is that it amounts to a dictator social choice function of the preferences of some Independent commissioner. In Section 4.3, we consider the same problem except with some number of unknown commissioners misevaluating the maps and having some arbitrary preference. In this case, choosing an Independent commissioner and choosing their highest-ranked preference as the map no longer works since they may have misevaluated the maps. With sufficiently many misevaluations, choosing the preference of the majority of Independents also fails.

Also, if there are no Independent commissioners, then it is not possible to just copy the preference profile of such a commissioner. Since the distribution of w_D and s_D and w_R and s_R are unspecified ahead of time, it is not possible to copy the preference profile of some w_D or w_R either. The mechanism described in Theorem 4.1 still applies in this setting.

4.2 The Unbalanced Mechanism

A natural question is what to do in a state where the underlying population has an unbalanced number of Democrats and Republicans. Is it fair to still force the commission to include the same number of both groups? Can we design a truthful mechanism that outputs neutral maps even with unbalanced numbers of commissioners? In Theorem 4.2, we generalize Theorem 4.1 to this setting. In doing so, we lose the anonymity of our mechanism, but this feels necessary to correct the obvious power imbalance between the commissioner groups. In Theorem 4.2, we assume that $n_D \geq n_R$, but the result also holds in the reverse setting by just swapping the roles of D and R throughout.

[3] We don't need to consider agents that prefer alternative N since N is winning already.

[4] Any outcome that would be preferred by s_D agents would be least preferred by s_R agents and vice versa, so we do not have to consider these groups jointly defecting from providing their true preferences. Hence, some subsets of the s_D's or s_R's colluding are the only groups we need to check for group strategy-proofness.

Theorem 4.2. *Let the number of Republican, Democrat, and Independent commissioners be given by the tuple (n_D, n_R, n_I) with $n_D \geq n_R$. Then, as long as there is at least one Independent, w_D, or w_R, the following type-dependent positional scoring rule is group strategy-proof and chooses a neutral map:*

$$D : \left\langle \frac{1}{n_D}, 0, -\frac{1}{n_D} \right\rangle, \ R : \left\langle \frac{1}{n_R}, 0, -\frac{1}{n_R} \right\rangle, \ I : \left\langle \frac{1}{n_R}, 0, -\frac{1}{n_R} \right\rangle.$$

Proof. As in Theorem 4.1, we let

$$\begin{aligned} x &\coloneqq \#s_D, \quad n_D - x = \#w_D; \quad \text{(from } D\text{)} \\ y &\coloneqq \#s_R, \quad n_R - y = \#w_R; \quad \text{(from } R\text{)} \\ z &\coloneqq \#w_D, \quad n_I - z = \#w_R. \quad \text{(from } I\text{)}. \end{aligned}$$

We now show that the mechanism outputs N when the agents report truthfully, and the mechanism is group strategy-proof. Firstly, if everyone is truthful, then N indeed wins since the scores for D, R, and N are

- D: $\frac{x}{n_D} - 1 - \frac{n_I - z}{n_R} \leq 0$;
- R: $\frac{y}{n_R} - 1 - \frac{z}{n_R} \leq 0$;
- N: $\frac{n_D - x}{n_D} + \frac{n_R - y}{n_R} + \frac{n_I}{n_R} > 0$,

using that $x < n_D$, $y < n_D$, or $n_I > 0$.

To prove that the mechanism is group strategy-proof, we first show that there is no useful deviation for s_D's.[5] To have a useful deviation for s_D's, we observe that the only possible useful deviation is misreporting $D \succ R \succ N$ again. Say there are x' of the s_D commissioners misreporting with $0 < x' \leq x$. Then, we have

- D': $\frac{x}{n_D} - 1 - \frac{n_I - z}{n_R}$;
- R': $\frac{y}{n_R} - 1 - \frac{z}{n_R} + \frac{x'}{n_D}$;
- N': $2 - \frac{x + x'}{n_D} - \frac{y}{n_R} + \frac{n_I}{n_R}$,

That is, $D' = D$, $R' = R + \frac{x'}{n_D}$, and $N' = N - \frac{x'}{n_D}$. Assume that $D' \geq R'$ since otherwise this is not a useful deviation. Hence, we have

$$\frac{x}{n_D} - 1 - \frac{n_I - z}{n_R} \geq \frac{y}{n_R} - 1 - \frac{z}{n_R} + \frac{x'}{n_D} \Rightarrow \frac{x - x'}{n_D} - \frac{n_I + y}{n_R} + \frac{2z}{n_R} \geq 0.$$

Finally, to have a useful deviation, we want $D' > N'$, i.e.,

$$\frac{x}{n_D} - 1 - \frac{n_I - z}{n_R} > 2 - \frac{x + x'}{n_D} - \frac{y}{n_R} + \frac{n_I}{n_R} \Leftrightarrow \frac{2x + x'}{n_D} - 3 + \frac{-2n_I + z + y}{n_R} > 0.$$

By adding $\frac{x - x'}{n_D} - \frac{n_I + y}{n_R} + \frac{2z}{n_R} \geq 0$, the above holds if

$$\frac{3x}{n_D} - 3 - \frac{3n_I}{n_R} + \frac{3z}{n_R} > 0 \Leftrightarrow \frac{x - n_D}{n_D} + \frac{z - n_I}{n_R} > 0 \lightning$$

[5] We still don't need to consider N since N is now winning already. However, we do need to consider R since it's asymmetric in this case.

hence there are no useful deviations for s_D's.

To show that there is no useful deviation for s_R's, we do a similar analysis. The only useful deviation for s_R's is to misreport $R \succ D \succ N$. Suppose there are y' of them misreporting with $0 < y' \leq y$. Then, we have

- D': $\frac{x}{n_D} - 1 - \frac{n_I - z}{n_R} + \frac{y'}{n_R}$;
- R': $\frac{y}{n_R} - 1 - \frac{z}{n_R}$;
- N': $2 - \frac{x}{n_D} - \frac{y+y'}{n_R} + \frac{n_I}{n_R}$.

Note that $D' = D + \frac{y'}{n_R}$, $R' = R$, and $N' = N - \frac{y'}{n_R}$. Assume that $R' \geq D'$ since otherwise this is not useful. T hen we have

$$\frac{y}{n_R} - 1 - \frac{z}{n_R} \geq \frac{x}{n_D} - 1 - \frac{n_I - z}{n_R} + \frac{y'}{n_R} \Rightarrow \frac{y - y'}{n_R} + \frac{n_I - 2z}{n_R} - \frac{x}{n_D} \geq 0.$$

Finally, to have a useful deviation, we want $R' > N'$, i.e.,

$$\frac{y}{n_R} - 1 - \frac{z}{n_R} > 2 - \frac{x}{n_D} - \frac{y + y'}{n_R} + \frac{n_I}{n_R} \Leftrightarrow \frac{2y + y'}{n_R} - 3 + \frac{-z - n_I}{n_R} + \frac{x}{n_D} > 0.$$

By adding $\frac{y-y'}{n_R} + \frac{n_I - 2z}{n_R} - \frac{x}{n_D} \geq 0$, the above holds if

$$\frac{3y}{n_R} - 3 + \frac{-3z}{n_R} > 0 \Leftrightarrow \frac{y - n_R}{n_R} - \frac{z}{n_R} > 0 \lightning$$

hence there are no useful deviations for s_R's. $\square$

4.3 Robustness

Given Theorem 4.2, we now consider the robustness of our mechanism by considering misevaluation of maps. For example, it is possible that for a commissioner's true preference is $D \succ N \succ R$ (i.e., they are of type s_D), but given three maps m_D, m_N, m_R, the commissioner instead reports $m_N \succ m_D \succ m_R$ due to the misevaluation on maps m_N and m_D. The goal is to have a theoretical guarantee that even with k commissioners misevaluating the maps (i.e., reporting wrong preferences), the result is still the neutral map.

Corollary 4.1. *Assuming $n_R \leq n_D$ and defining x, y, and z as in Theorem 4.2, the mechanism in Theorem 4.2 is robust up to k misevaluations for any*

$$k < \frac{1}{4}\left(n_I + 2n_R - \frac{n_R}{n_D} \cdot x - y\right).$$

In particular, $k < n_I/4$.

Proof. Recall that in the truthful reporting setting we have the scores for D, R, and N being

- D: $\frac{x}{n_D} - 1 - \frac{n_I - z}{n_R} \leq 0$;
- R: $\frac{y}{n_R} - 1 - \frac{z}{n_R} \leq 0$;

- N: $\frac{n_D-x}{n_D} + \frac{n_R-y}{n_R} + \frac{n_I}{n_R} > 0$,

Suppose $k < (n_I + 2n_R - \frac{n_R}{n_D} \cdot x - y)/4$ people misreport. We observe that any score can go up or down by at most $\frac{2k}{n_R}$, i.e., Theorem 4.2 is robust for

$$N \text{ score} > \max(D \text{ score}, R \text{ score}) + \frac{4k}{n_R}.$$

This amounts to checking the following two cases hold.

- Case 1: N score $-$ D score $- \frac{4k}{n_R} > 0$. This is because

$$\begin{aligned}\frac{n_D - 2x}{n_D} + \frac{n_R - y}{n_R} + 1 + \frac{2n_I - z}{n_R} - \frac{4k}{n_R} &\geq \frac{2n_D - 2x}{n_D} + \frac{n_R - y}{n_R} + \frac{n_I - 4k}{n_R} \\ &= \underbrace{\frac{n_D - x}{n_D}}_{\geq 0} + \underbrace{\frac{n_I + 2n_R - \frac{n_R}{n_D} \cdot x - y - 4k}{n_R}}_{>0}.\end{aligned}$$

- Case 2: N score $-$ R score $- \frac{4k}{n_R} > 0$. This is because

$$\begin{aligned}\frac{n_D - x}{n_D} + \frac{2n_R - 2y}{n_R} + \frac{n_I + z}{n_R} - \frac{4k}{n_R} &= \frac{n_D - x}{n_D} + \frac{2n_R - 2y}{n_R} + \frac{z}{n_R} + \frac{n_I - 4k}{n_R} \\ &= \frac{n_R - \frac{n_R}{n_D} \cdot x}{n_R} + \frac{2n_R - 2y}{n_R} + \frac{z}{n_R} + \frac{n_I - 4k}{n_R} \\ &= \underbrace{\frac{n_R - y + z}{n_R}}_{\geq 0} + \underbrace{\frac{n_I + 2n_R - \frac{n_R}{n_D} \cdot x - y - 4k}{n_R}}_{>0}.\end{aligned}$$

□

Corollary 4.1 has the following nice interpretation.

Remark 2. *Note that $n_I + 2n_R - \frac{n_R}{n_D} \cdot x - y$ is exactly*

$$n_I + \frac{n_R}{n_D} \cdot \#(w_D \textit{ from } D \textit{ commissioners}) + \#(w_R \textit{ from } R \textit{ commissioners}).$$

That is, every commissioner who prefers a neutral map to a biased map contributes to the robustness of this mechanism.

5 Discussion and Conclusion

In this paper, we provide an intuitive model of citizens redistricting commissions selecting a redistricting map, with the goal being the selection of a map without partisan bias. We assume that only the commissioners know the true partisan bias of maps and that the preferences of each commissioner from each partisan group are only known to be of one of two types. We show that, under this model, there exists a group strategy-proof mechanism for eliciting commissioner preferences and outputting a neutral map. This mechanism has other desirable qualities: it works for commissions with an unbalanced number of Republican and Democrat commissioners, and it is also robust to some commissioners misevaluating the partisan bias of the maps (and hence accidentally misreporting their preferences).

Extensions of this work could go in several different directions. First, what if we considered more forms of map bias than just partisan bias? The result would be more complicated commissioner preference profiles and a less obvious notion of a fair map. Nonetheless, it would be interesting to investigate the settings in which there exists a robust, group strategy-proof mechanism outputting a fair map.

One of the most famous existing strategy-proof voting rules is the median voting rule. The median voting rule applies in preference domains in which the agents all have single-peaked preference profiles.

Open Question 1. *Is there a reasonable model of redistricting in which the space of redistricting maps can be viewed as a single-peaked preference domain?*

Another avenue is to carefully analyze the commissioner selection process. Currently, most independent redistricting commissions elicit applications and randomly select from a pool of commissioners via their professed partisan affiliations. There are safety measures in place to disqualify applicants who have existing ties to politics, but the incentive structure of this process warrants careful analysis.

Open Question 2. *How should the applicant selection process for independent citizens redistricting commissions be organized to incentivize the creation of a diverse, representative, and just commission?*

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