GraSS: Scalable Influence Function with Sparse Gradient Compression

A Foray to **Efficient** Data Attribution and Influence Function

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Table of Content

Table of Content

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Example (Running example)

We will consider the classical Influence Function [KL17] throughout the talk.

Table of Content

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Popular choice of B: $B_i = \{z_i\}$ for $z_i \in D$, i.e., $\tau_f(B_i)$ provides the *point-wise* effect.

Intuition (Estimating τ_f)

Parametrize D by a default weight vector $w = \mathbb{1}/n \in \mathbb{R}^n$ for the data points z_i's.

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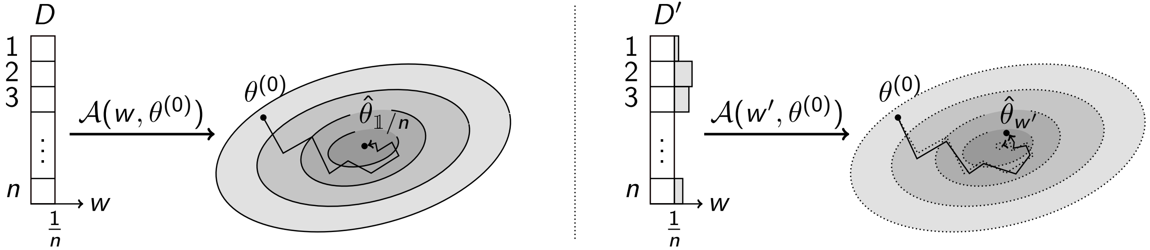

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Influence Function

Theorem (Influence function [KL17; Gro+23])

Let $\hat{\theta} = \hat{\theta}_{\mathbb{1}/n}$ *be the ERM trained on* D *and* $H_{\hat{\theta}} = \frac{1}{n}\sum_{z_i \in D} \nabla_{\theta}^2 \ell_i$ *be the empirical Hessian.*

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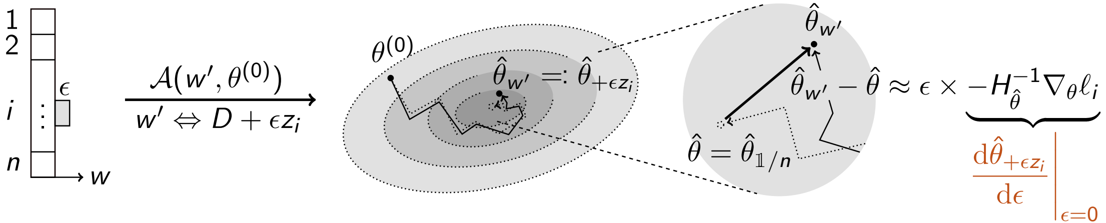

Table of Content

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As previously seen (Influence function)

Counterfactual prediction of removing z_i is $\Delta f = \tau_f(\{z_i\}) \approx \epsilon \cdot \mathcal{I}(z_i, f)$ with $\epsilon = -1/n$, where

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- Inverse-Hessian $H_{\hat{\theta}}^{-1} \in \mathbb{R}^{p \times p}$: inverting a $p \times p$ second-order Hessian

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- ***Computation***: *inverse-Hessian* $O(p^3)$
- ***Storage***: *vectors* + *inverse-Hessian* $O(np + p^2)$

Table of Content

Classical iHVP

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Remark

iHVP in influence function specifically is different and orthogonal to above.

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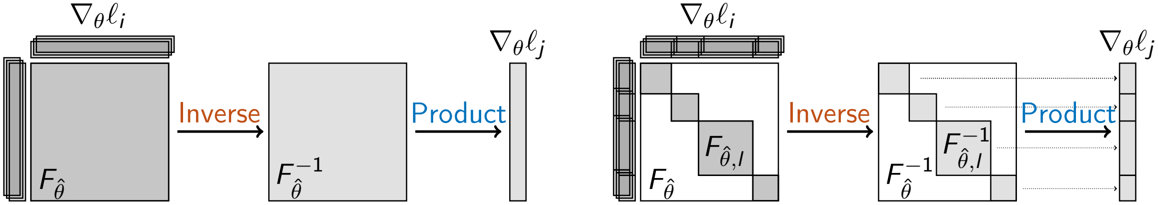

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$\Rightarrow$ Replacing p with k everywhere (with some **computation** overhead...)

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We can compress $g_i := \nabla_\theta \ell_i \in \mathbb{R}^p$ *down to* $\widetilde{g}_i \in \mathbb{R}^k$ *for some* $k \ll p$.

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- but for us, we can also compress ∇f and take inner product without problems!

Example (Gaussian/Rademacher Projection (Random [Woj+16]))

Linear map induced by $P \in \mathbb{R}^{k \times p}$ with $P_{ij} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ or $\mathcal{U}(\{\pm 1\})$ satisfies the JL lemma.

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Random and its Computational Complexity

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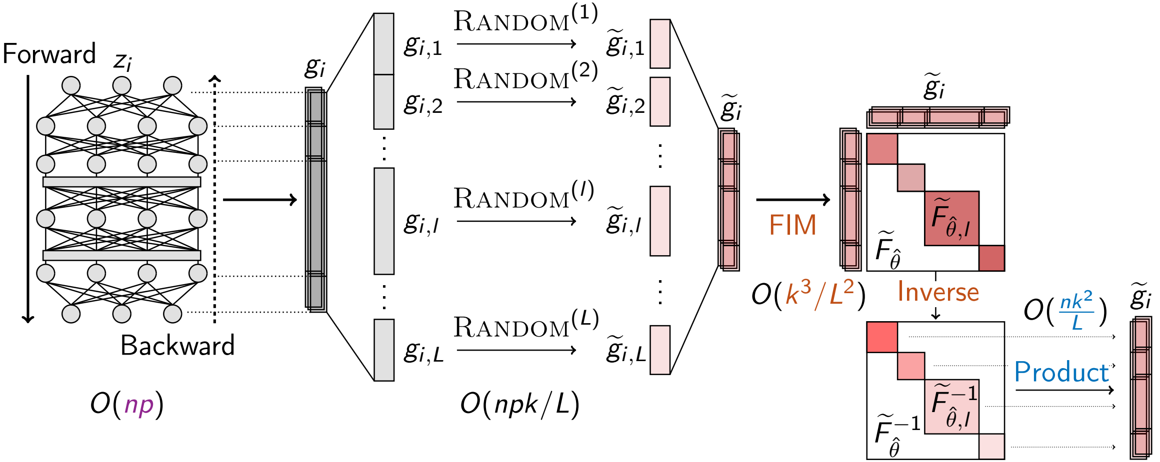

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Problem

How to speed up the overhead of compression?

A natural idea is to search for faster compression algorithm:

[3] This is also used in TRAK's implementation (https://github.com/MadryLab/trak).

Fast Johnson-Lindenstrauss Transform

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In total, for all data points and all layers, FJLT takes $O(n(p+k)\log p)$

Remark

It's roughly the same for one training epoch!

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Putting Everything Together: FJLT

Stage 0	Stage 1	Stage 2
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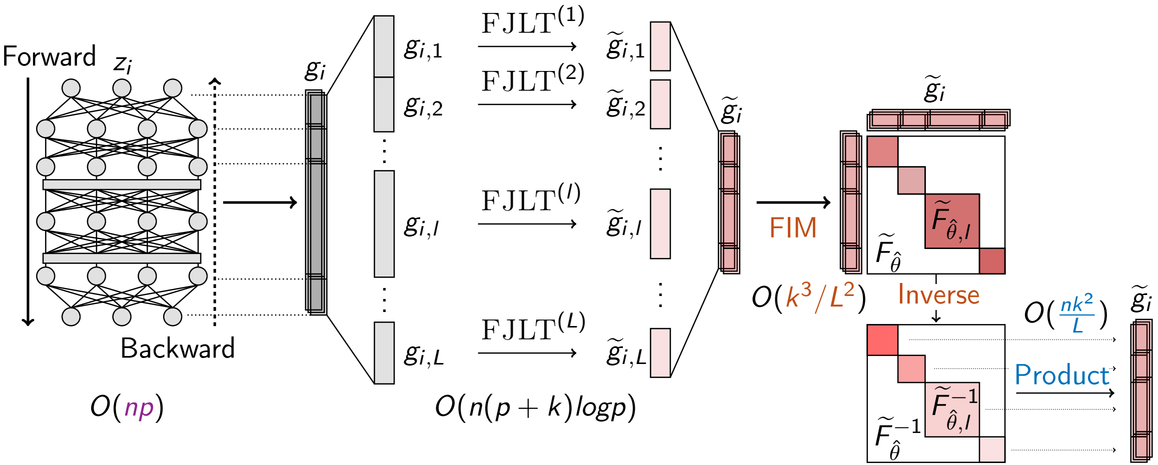

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- **Dense Matrix**: Each entry of $P^{(l)}$ is sampled i.i.d. from $\mathcal{U}(\{\pm 1\})$.
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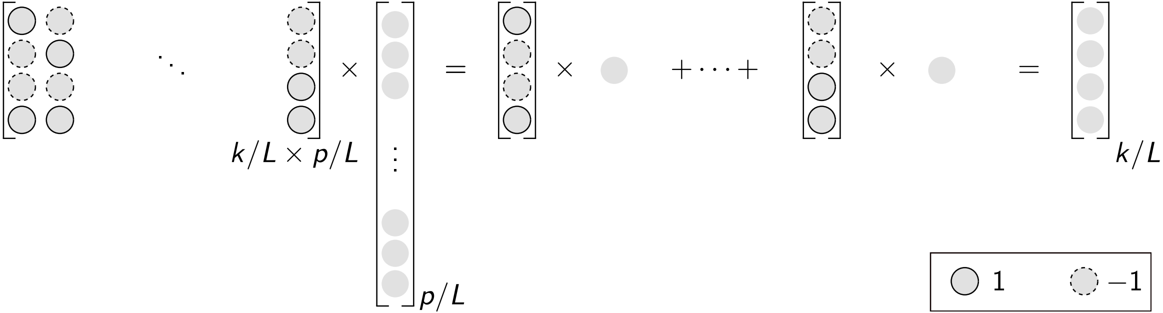

Sparse Johnson-Lindenstrauss transform [DKS10; KN14] considers a **sparser** $P^{(l)}$ instead:

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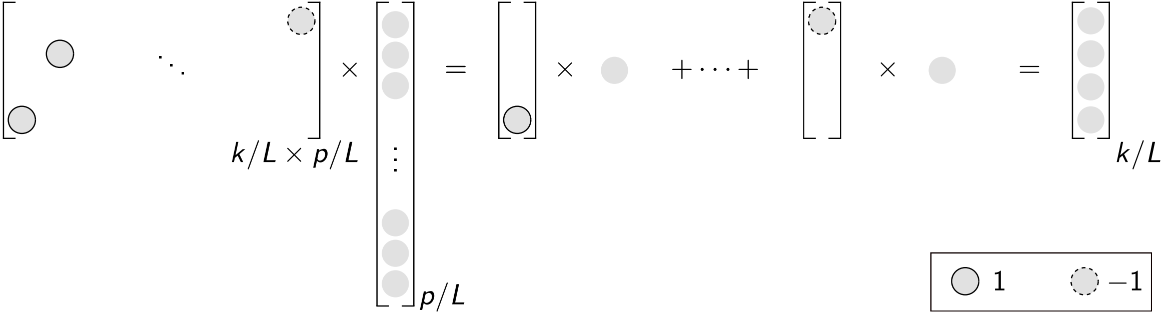
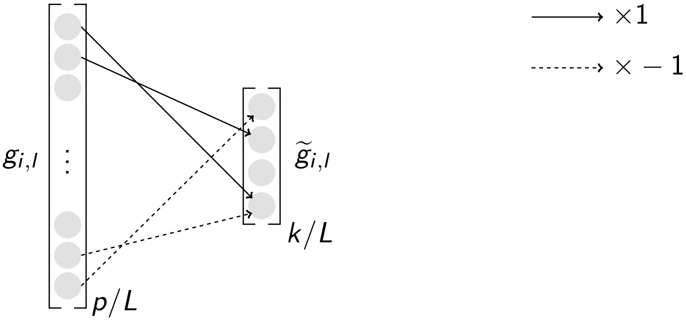


SJLT: Alternative Viewpoint

Equivalently, you can think about SJLT as follows:



Intuition

For each entry of $g_{i,l}$, we select s entries in $\widetilde{g}_{i,l}$ to add on (or subtract from, depending on ± 1).

Computational Complexity of SJLT

SJLT only depends on input dimension p/L:

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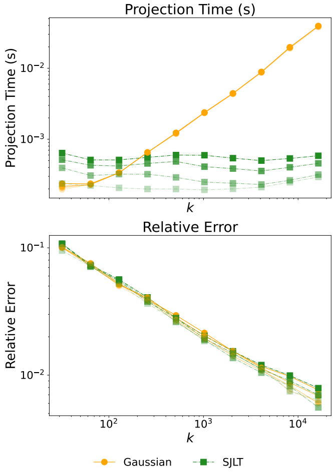
Remark (Potential speedup)

SJLT *exploits input sparsity, each runs only in* $O(\mathrm{nnz}(g_{i,l}))$.

- *Potentially,* SJLT *can run faster than* $O(np)$ *in total.*



$p = 131{,}072$ on several sparsity levels[4]

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Compression via selecting a few parameters ($\Leftrightarrow$ masking out most parameters):

Intuition

Instead of "compress everything succinctly," we select a few parameters to look at.

- In the literature, people find out that only a few parameters are important for "inference"
- Idea of *localization* emerges [He+25; Yad+23; Wan+24].
- Used for task merging, sparsification, etc.

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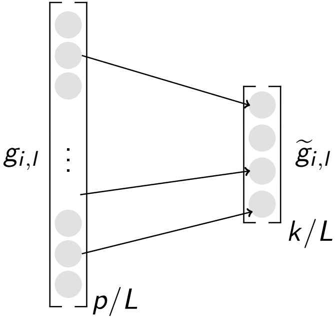

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We finally achieve sub-linear *compression:*

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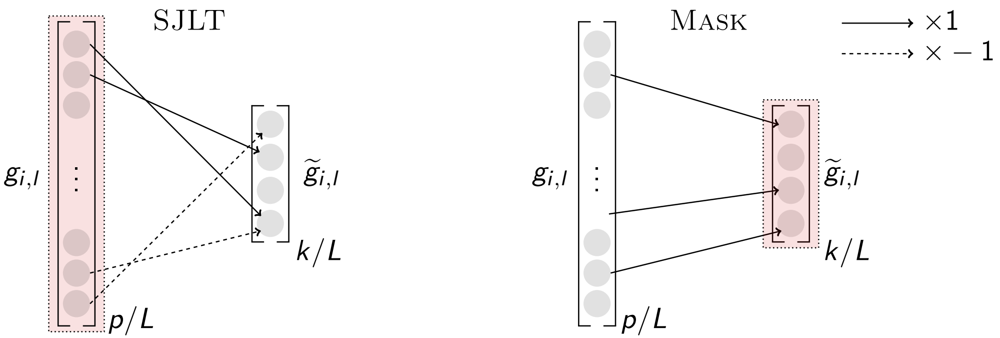
This complexity should now be impossible to beat.

Problem

In what cost?

Situation Now

We now have two candidates, SJLT and MASK:



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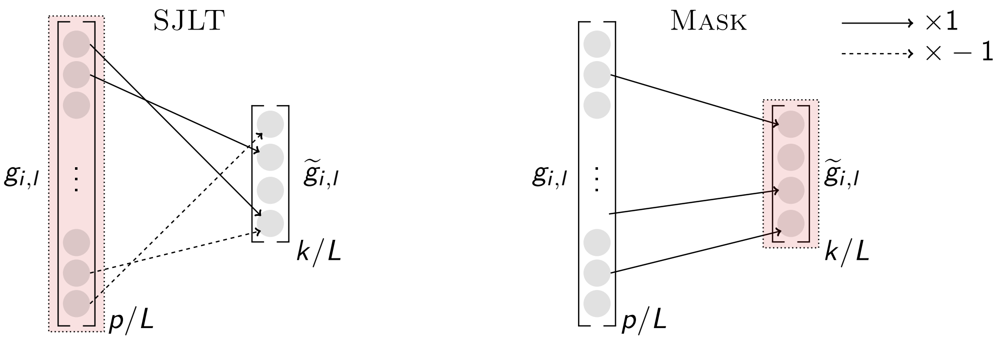

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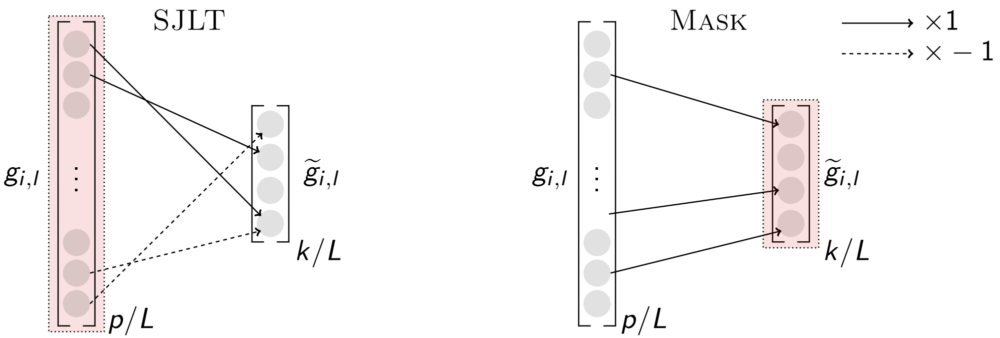
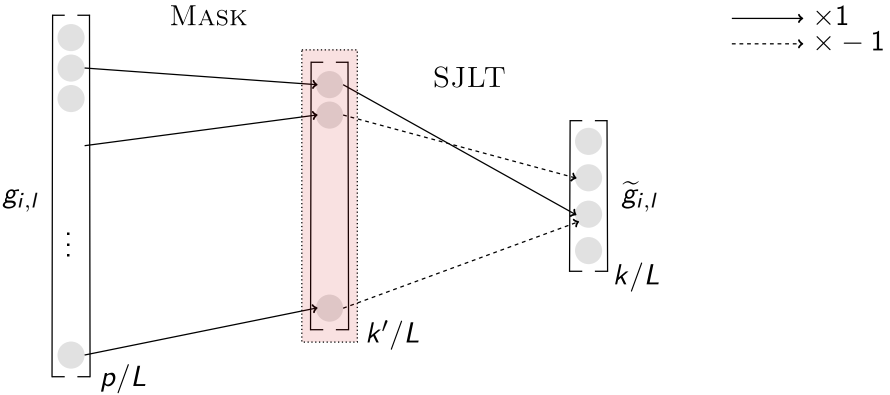


Problem (Pros and Cons)

- SJLT: *Very good compression guarantees, but cost $\propto$ input dimension.*
- MASK: *Extremely fast with cost $\propto$ output dimension, but will lose a lot of information.*



Intuition

First Mask *to a moderate dimension k'/L, then* SJLT *to the final dimension k/L!*

We term this method as GraSS: **Gra**dient **S**parsification and **S**parse projection.

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In total, for all data points and all layers, GraSS takes $O(nk')$.

Putting Everything Together Again

Let's put everything together again, this time with GraSS.

Stage 0: Compute all per-sample gradients $g_i \in \mathbb{R}^p$

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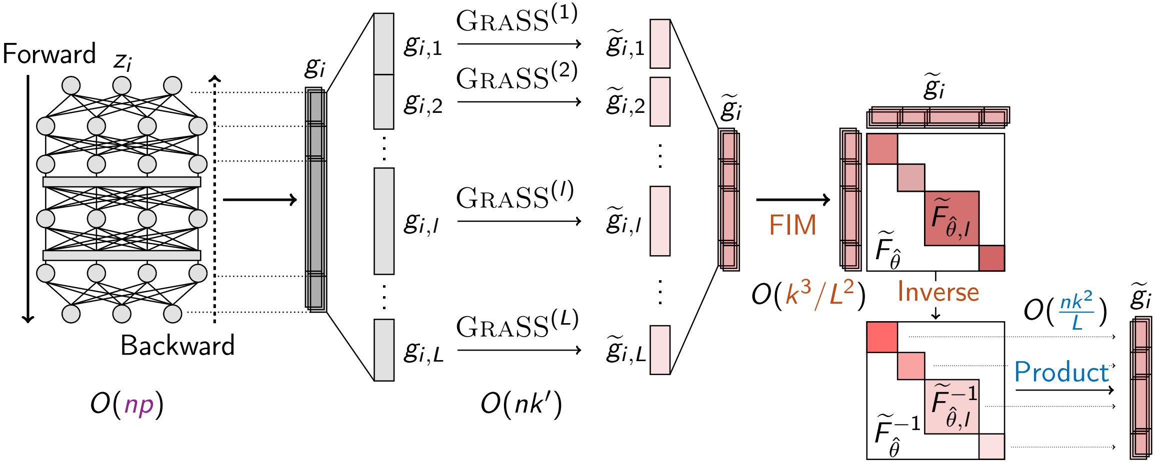

Table of Content

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We will see their fundamental ideas next. Let's first recall some basic facts about linear layers.

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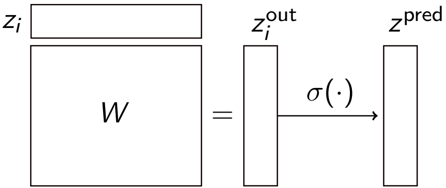

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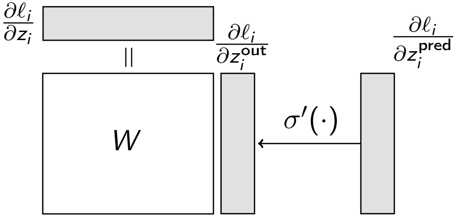

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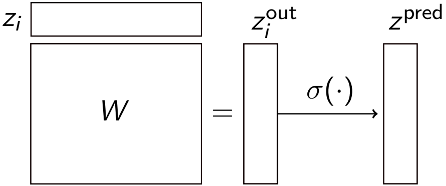

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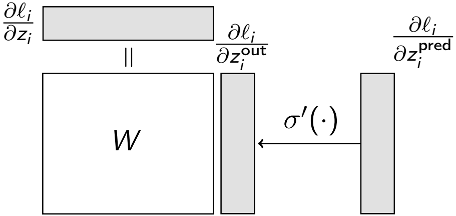

Remark

What we actually want is g_i:

$$g_i = \frac{\partial \ell_i}{\partial W} = \frac{\partial \ell_i}{\partial z_i^{out}} \frac{\partial z_i^{out}}{\partial W} = z_i \otimes \frac{\partial \ell_i}{\partial z_i^{out}}$$

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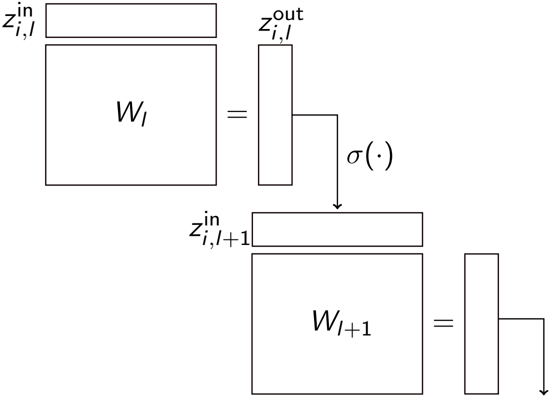

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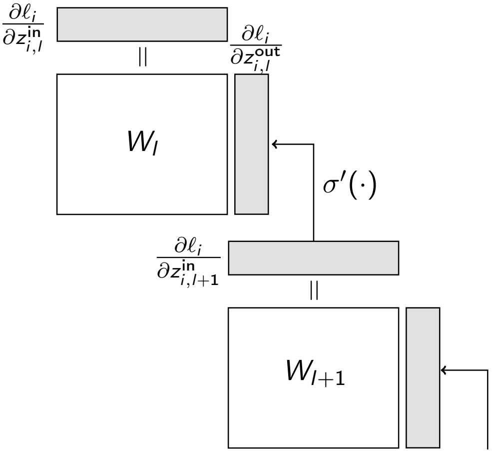

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We should now see the problem:

Problem

In the computational graph, we never materialize $g_{i,l}$.

Materializing Gradients

Remark

What we actually want:

$$g_{i,l} = \frac{\partial \ell_i}{\partial W_l} = \frac{\partial \ell_i}{\partial z_{i,l}^{out}} \frac{\partial z_{i,l}^{out}}{\partial W_l} = z_{i,l}^{in} \otimes \frac{\partial \ell_i}{\partial z_{i,l}^{out}}$$

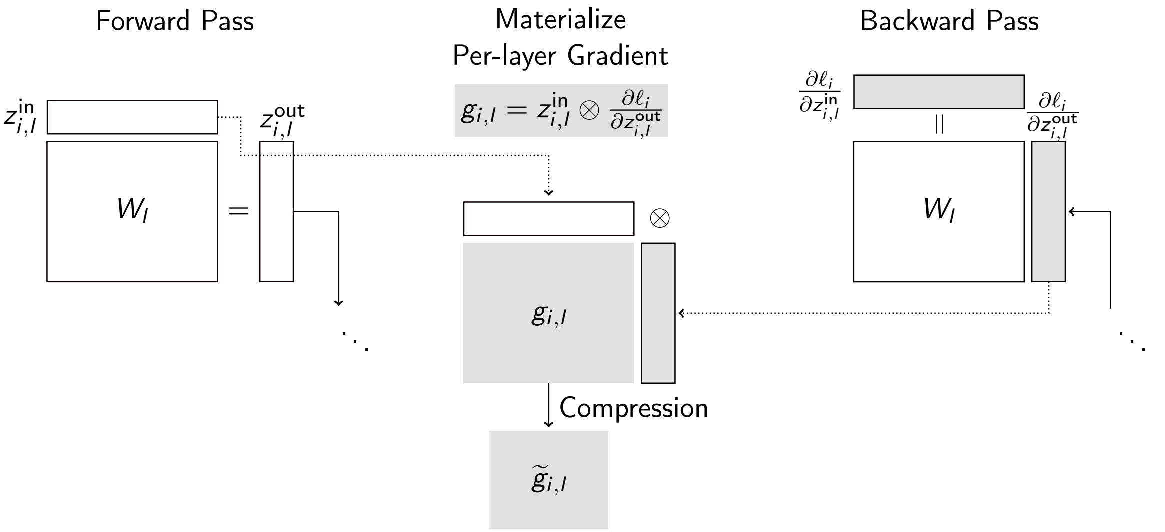
We should now see the problem:

Problem

In the computational graph, we never materialize $g_{i,l}$.

Hence, our previous analysis neglects the cost of computing $g_{i,l}$!

Materializing Gradients



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Even if GRASS *takes only $O(nk') \ll O(np)$, once we materialize $g_{i,l}$, it'll take $O(np)$.*

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Even if GRASS *takes only* $O(nk') \ll O(np)$, *once we materialize* $g_{i,l}$, *it'll take* $O(np)$.

However, is this really a concern?

- I mean, how can you compress $g_{i,l}$ without materializing it?
- Seems like this $O(np)$ cost will lay in the background and we can't get rid of?

Putting Everything Together for Linear Layers

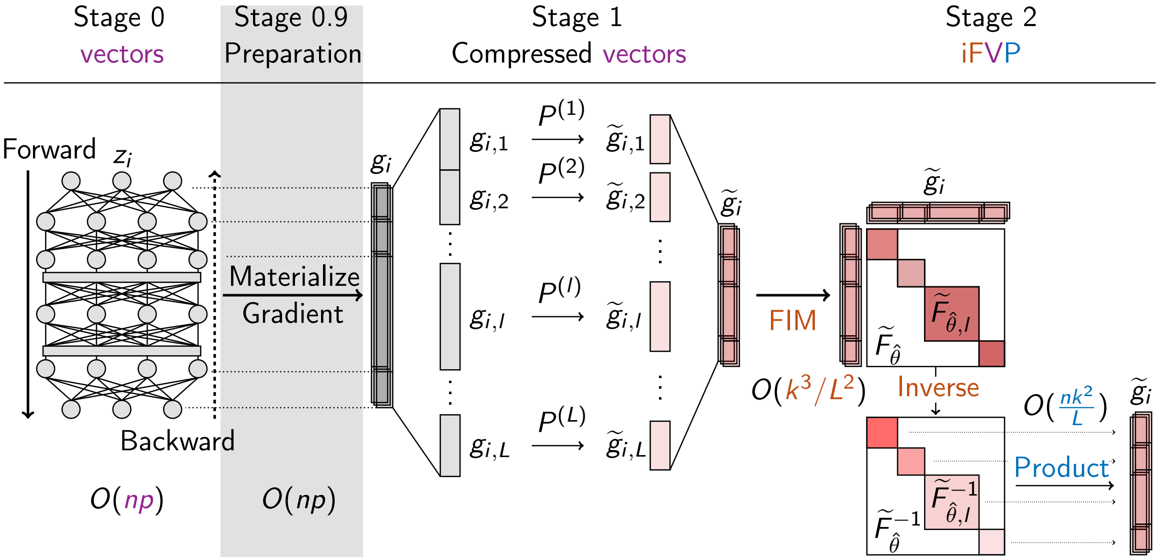

Table of Content

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Theorem (LoGra)

There is a gradient compression algorithm that ***does not*** *require materializing* $g_{i,l}$ *(for MLP layer).*[5]

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- *Allocating k/L equally $\Rightarrow$ target dimension for both is $\sqrt{k/L}$*

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As previously seen (LoGra)

$$\widetilde{g}_{i,l} = P^{(l)} g_{i,l} = (P_{in}^{(l)} \cdot z_{i,l}^{in}) \otimes \left(P_{out}^{(l)} \cdot \frac{\partial \ell_i}{\partial z_{i,l}^{out}} \right)$$

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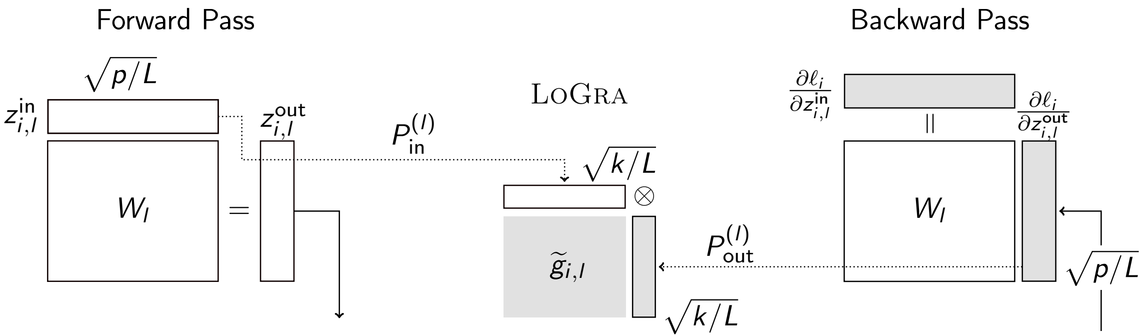

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Overall, LoGra only takes $O(n\sqrt{kp}) < O(np)$

Putting Everything Together: LoGra

Stage 0	Stage 0.9	Stage 1	Stage 2
vectors	Preparation	Compressed vectors	iFVP

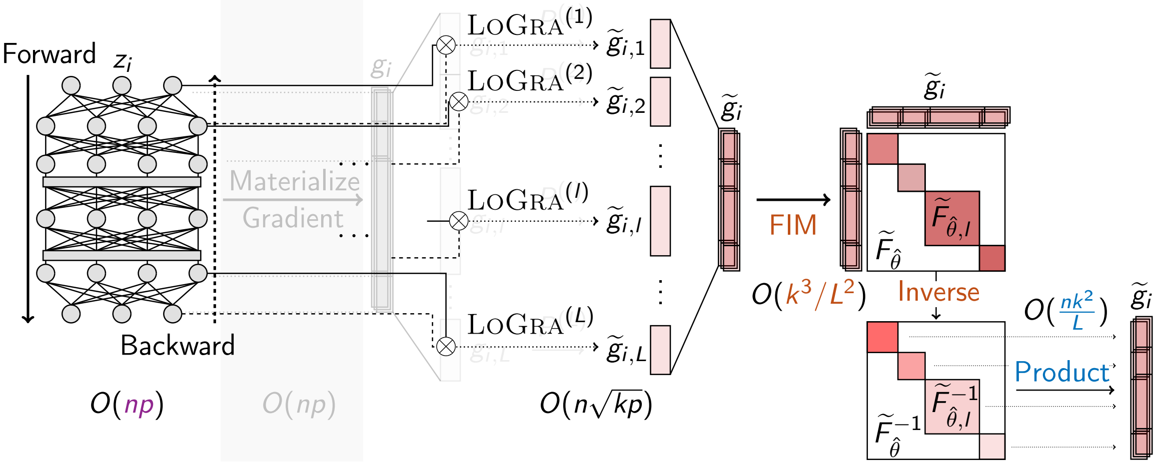

Table of Content

Now What?

Let's summarize the situation a bit. For *general layers*:

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Problem

How to beat LoGra*?*

Naive Approach

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Intuition

Apply SJLT *to a moderate dimension!*

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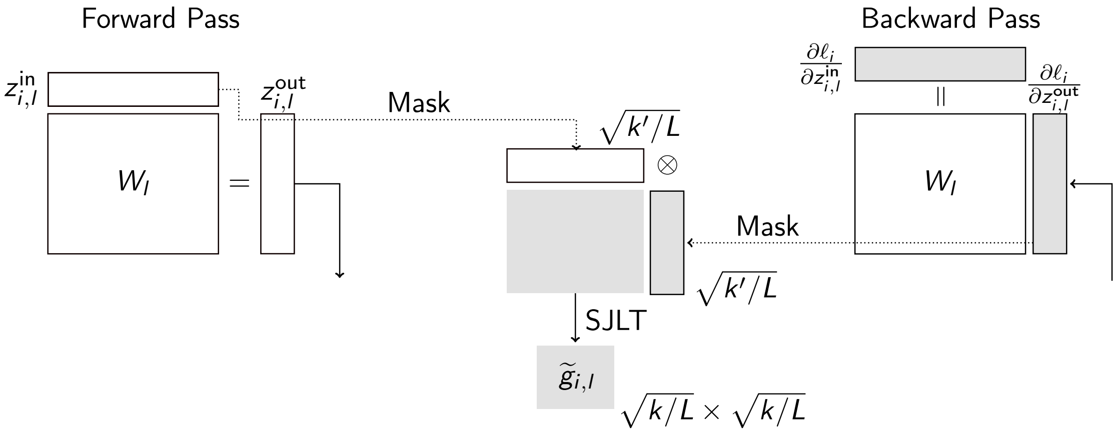

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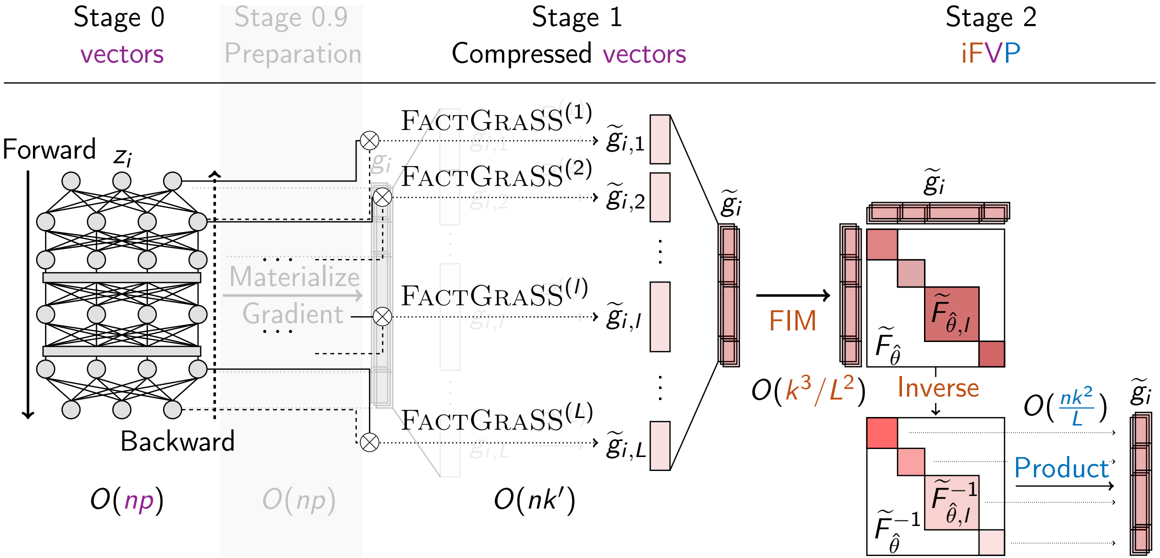
Overall, FactGraSS takes $O(nk')$, same as GraSS, *but without materializing* $g_{i,l}$!

Putting Everything Together: FactGraSS



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Remark

Compared to LoGra which takes $O(n\sqrt{kp})$, FactGraSS is faster when

$$nk' < n\sqrt{kp} \Leftrightarrow k' < \sqrt{kp}.$$

Let $k' = ck$, then above is equivalent to $ck \leq \sqrt{kp} \Leftrightarrow c \leq \sqrt{p/k}$.

Table of Content

Experimental Setup

We consider the following setups:

- experiment on TRAK and influence function
- focus on *speed* and *accuracy* of our method

Quantitative Study: Small model and datasets

- Accuracy: Able to measure *LDS scores*
- Efficiency: Compare *wall-time* difference for projection

Qualitative Study: Large model and datasets

- Accuracy: Case study on the most influential data points
- Efficiency: Focus on *throughput*

Table of Content

Quantitative Study

	Sparsification			Sparse Projection			Baselines					
	$\textsc{Mask}_k$			SJLT_k			FJLT_k			$\textsc{Random}_k$		
k	2048	4096	8192	2048	4096	8192	2048	4096	8192	2048	4096	8192
LDS	0.3803	0.4054	0.4318	**0.4171**	**0.4280**	**0.4357**	0.4146	0.4359	0.4347	0.4101	0.4253	0.4346
Time (s)	**0.1517**	**0.1458**	**0.1501**	0.4919	0.5172	0.4754	0.8997	1.4341	2.4387	3.0806	5.5421	10.8355

Table: MLP with MNIST on TRAK.

	Sparsification			Sparse Projection			GraSS			Baseline		
	$\textsc{Mask}_k$			SJLT_k			$\text{SJLT}_k \circ \textsc{Mask}_{4k_{\text{max}}}$			FJLT_k		
k	2048	4096	8192	2048	4096	8192	2048	4096	8192	2048	4096	8192
LDS	0.3690	0.4116	0.4236	0.4131	**0.4499**	0.4747	0.4123	0.4357	0.4545	**0.4157**	0.4497	**0.4753**
Time (s)	**0.1026**	**0.1074**	**0.1296**	12.3590	12.2393	17.4836	0.3652	0.3648	0.3993	31.5491	48.1669	81.9322

Table: ResNet9 with CIFAR2 on TRAK.

Quantitative Study

	Sparsification			Sparse Projection			GraSS			Baseline		
	Mask_k			SJLT_k			$\text{SJLT}_k \circ \text{Mask}_{64k_{\max}}$			FJLT_k		
k	2048	4096	8192	2048	4096	8192	2048	4096	8192	2048	4096	8192
LDS	0.1281	0.1456	0.1469	**0.3062**	0.3533	0.3861	0.2840	0.3242	0.3413	0.2907	**0.3585**	**0.4011**
Time (s)	**0.5341**	**0.5067**	**0.5179**	21.6460	21.1881	21.3192	2.6934	2.6071	2.7202	100.8136	156.0613	269.9093

Table: MusicTransformer with MAESTRO on TRAK.

	Sparsification			Sparse Projection			FactGraSS			Baseline (LoGra)		
	$\text{Mask}_{\sqrt{\hat{k}} \otimes \sqrt{\hat{k}}}$			$\text{SJLT}_{\sqrt{\hat{k}} \otimes \sqrt{\hat{k}}}$			$\text{SJLT}_{\sqrt{\hat{k}}^2} \circ \text{Mask}_{2\sqrt{\hat{k}} \otimes 2\sqrt{\hat{k}}}$			$\text{Random}_{\sqrt{\hat{k}} \otimes \sqrt{\hat{k}}}$		
$\hat{k}$ $(= k/L)$	256	1024	4096	256	1024	4096	256	1024	4096	256	1024	4096
LDS	0.1034	0.1479	**0.2391**	**0.1240**	**0.1897**	0.2389	0.1126	0.1784	0.2360	0.1188	0.1818	0.2338
Time (s)	**5.4933**	**5.3643**	**5.6385**	132.5404	133.4029	136.5163	6.5790	7.4161	6.3075	20.4839	20.9835	22.2157

Table: GPT2-small with WikiText on (block-diagonal FIM) influence function.

Table of Content

Qualitative Study

Next, we compare FactGraSS and LoGra on billion-scale model and dataset

$\hat{k}$ $(= k/L)$	Compress			iHVP		
	256	1024	4096	256	1024	4096
LoGra	27,292	27,255	26,863	7,307	7,478	7,367
FactGraSS	**72,218**	**72,684**	**73,811**	**8,584**	**8,594**	**8,681**

Table: Throughput (tokens/s) for Llama-3.1-8B-Instruct on (block-diagonal FIM) influence function.

Remark

In terms of gradient compression, FactGraSS *outperforms* LoGra *by 160%.*

Qualitative Study

To improve data privacy,

To improve data privacy, the European Union has implemented the General Data Protection Regulation (GDPR). ...
Data Protection Principles
The GDPR sets out six data protection principles...

- **Lawfulness, fairness, and transparency**: Businesses must process personal data in a way that is lawful, fair, and transparent. ...
- **Storage limitation**: Businesses must not store personal data for longer than necessary. ...

Data Subject Rights
The GDPR gives individuals a range of rights when it comes to their personal data. These rights include:

- **Right to access**: Individuals have the right to access their personal data and obtain information about how it is being processed. ...
- **Right to erasure**: Individuals have the right to have their personal data deleted if it is no longer necessary for the purposes for which it was collected. ...

Influential Data

...
The fact of registration and authorization of users on Sputnik websites via users' account or accounts on social networks indicates acceptance of these rules.
Users are obliged abide by national and international laws. ... The administration has the right to delete comments made in languages other than the language of the majority of the websites ...
...

- violates privacy, distributes personal data of third parties without their consent or violates privacy of correspondence; ...
- pursues commercial objectives, contains improper advertising unlawful political advertisement or links to other online resources ...

The administration has the right to block a user's access to the page or delete a user's account without notice if the user is in violation of these rules or if behavior indicating said violation is detected.
If the moderators deem it possible to restore the account/unlock access, it will be done. In the case of repeated violations of the rules above resulting in a second block of a user account, access cannot be restored. ...

Q&A Time!

Thanks! Ask ***anything*** you want!

[ABH17] Naman Agarwal, Brian Bullins, and Elad Hazan. "Second-order stochastic optimization for machine learning in linear time". In: *Journal of Machine Learning Research* 18.116 (2017), pp. 1–40.

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